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Tour Planning for an Unmanned Air Vehicle Under Wind Conditions

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A very important subproblem in the task assignment problem for unmanned air vehicles is the evaluation of costs for the state transitions of a directed graph. Usually a Dubins vehicle flying in the absence of wind is considered in the computation of costs. However, when a prevailing wind vector field is considered, the costs taken on very different values and the task assignment problem can have very different solutions. In this paper, we consider the problem of constructing minimum-time trajectories for a Dubins vehicle in the presence of a time-varying wind vector field. We present results on the existence and uniqueness of minimum-time solutions for a Dubins vehicle flying in a general time-varying wind vector field under some technical conditions. These results extend the conclusions of the well-known Dubins theorem. We also propose an algorithm for obtaining the minimum-time solution for an unmanned air vehicle and prove its convergence. We also present the results of numerical experiments that show that the importance of considering wind vector fields while planning the tour for an unmanned air vehicle.

Nomenclature

q	=	(x, ψ)
t	=	time in seconds
t_f	=	free final time for trajectory planning problem
$u(\cdot)$	=	steering control input to the MAV
$u_{\max}$	=	upper bound on $\lvert u(t)\rvert$ for all t
V	=	constant speed of the MAV
$W(\cdot,\cdot)$	=	wind vector field is a function of x and t in general
x	=	position of center of mass of the MAV in IR^2
$(x_f, \psi_{\mathrm{fd}})$	=	desired position and heading at final time t_f
(x_0, ψ_0)	=	initial position and heading at initial time 0
z	=	position of the MAV after a coordinate transformation for numerical stability
θ	=	heading of the MAV after a coordinate transformation
ϕ	=	angle by which the initial coordinate system is rotated for numerical stability of the trajectory planning algorithm
ψ	=	heading of the MAV (in radians)

Introduction

COOPERATIVE control of multiple, autonomous unmanned air vehicles (UAVs) is an active area of research that holds enormous potential for military and civilian applications [1–3]. This new paradigm for control has been implemented in the MultiUAV simulation platform by the U.S. Air Force Research Laboratory [3]. It has a hierarchical architecture, where at the highest level the dynamics of the controlled agents are usually suppressed and a task allocation for the agents is performed using graph theory. The tasks for the agents are usually tightly coupled in time [3,4] and hence estimates of the times taken for each agent to fly from one destination to the next are highly critical for correct assignment of tasks. This estimate of times is usually obtained by considering a kinematic model of the air vehicle, along with kinematic turn-radius constraints, to keep the computation time at a manageable level [5–7]. The key result that is used in this computation is Dubins theorem on the existence of minimum-time solutions for a kinematic model with minimum turn-radius constraint [8]. However, this result is only valid for zero wind and hence all of the available cost estimation algorithms are only valid for zero wind (except for [11]). In this paper, we propose a new method for the computation of minimum-time solutions for the trajectory planning problem for a micro air vehicle (MAV) or UAVs in the presence of time-varying wind and prove its convergence. We also present the results of numerical experiments that show that the order in which targets need to be overflown to minimize total flight time can be very different if the winds are accounted for.

We model an MAV flying with a constant speed in the wind axes and at a constant altitude. The kinematic equations of motion for the MAV are

$$\dot{x} = V(\cos\psi, \sin\psi) + W(x,t); \qquad \dot{\psi} = u \tag{1}$$

where $x = (x_1, x_2)$, $W(x,t) = (W_1(x,t), W_2(x,t))$, and $W_i(x,t)$; $i = 1, 2$ are functions with bounded derivatives. These equations contain the wind vector field that is not considered in earlier works [5–9]. Of course, the actual model for a MAV consists of Euler's equations [10] and the kinematic model used above implies a perfect response to the turn commands. However, it must be kept in mind that the kinematic model is only employed to compute the order in which the targets are to be visited. Once the order is determined, optimal control theory should be employed to determine the actual time-optimal path to be flown by the aircraft. A major reason for using the kinematic model is the fact that only necessary conditions for optimality exist for the second order model (given by Pontryagin's maximum principle) and no results on the existence of solutions are available. The situation is different for the kinematic model where in the absence of wind [that is $W(x,t) = (0,0)$], the well-known Dubins's theorem [8] posits the existence of a time-optimal solution for any combination of initial and final positions and velocities of the UAV or MAV.

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Contemporaneously during the development of the results in this manuscript, McGee et al. [11] have addressed the problem of trajectory planning in the presence of *constant* wind. The mathematical method employed by them appears similar to ours for the case of constant wind. They classify the set of initial and final states into types 1 and 2, where type 1 states have a minimum-time solution while type 2 states do not. Here we prove that their type 2 states form a set of measure zero which means that they are sparse in the set of all possible states. In our earlier work [12], we addressed the more general case of time and space-varying winds and proved the existence of time-optimal solutions (for almost every initial and final states) under technical conditions involving the gradients of the wind. The verification of these conditions can be used as the starting point for a numerical algorithm to compute the time-optimal trajectory. Here, we consider time-varying wind vector fields [that is $W(x,t) = W(t)$ for all $x \in \mathrm{IR}^2$ and any $t \in \mathrm{IR}$] and prove the convergence of an algorithm for computing the minimum-time solutions. Therefore, the work in this manuscript is a generalization of the work of McGee et al. [11].

Getting back to the equations of motion, let $q = (x, \psi)$. The optimal control problem to be solved can be formulated in two ways:

Problem 1: For the system (1), minimize the final time t_f subject to the constraints: $q(0) = q_0 = (x_0, \psi_0)$ and $q(t_f) = q_f = (x_f, \psi_{\mathrm{fd}})$; $u(\cdot)$ is a Lebesgue measurable function, and satisfies

$$|u(t)| \le u_{\max} = \frac{V}{R_{\min}} \tag{2}$$

for all t. In Eq. (2), $R_{\min}$ is the minimum turn radius *in the absence of wind* and arises due to mechanical limitations on the aircraft. As an MAV is designed to fly between the streets while searching for targets, we must ensure that the MAV enters a street along its centerline, or at least parallel to it. In our numerical examples, ψ_{fd} is the angle that the centerline makes with the x axis of the coordinate system. Because of the wind vector field, the actual tangent line to the solution of Problem 1 would make an angle:

$$\begin{aligned}\angle \dot{x}(t_f) &= \arctan\left(\frac{V \sin\psi(t_f) + W_2(x, t_f)}{V\cos\psi(t_f) + W_1(x,t_f)}\right)\\ &= \arctan\left(\frac{V\sin\psi_{\mathrm{fd}} + W_2(x,t_f)}{V\cos\psi_{\mathrm{fd}} + W_1(x,t_f)}\right)\end{aligned} \tag{3}$$

which could result in the loss of the MAV. Hence, it is clear that one must change the end condition to require that $\angle\dot{x}(t_f) = \psi_{\mathrm{fd}}$ to minimize the possibility of aircraft loss. We denote $\psi(t_f)$ by ψ_f for simplicity and also denote $W_1(x,t_f) = \|W(x_f,t_f)\|_2 \cos\theta_W(x_f,t_f)$ and $W_2(x,t_f) = \|W(x_f,t_f)\|_2 \sin\theta_W(x_f,t_f)$. The following computations yield a formula for ψ_f:

$$\frac{V\sin\psi_f + W_2(x,t_f)}{V\cos\psi(t_f) + W_1(x,t_f)} = \tan\psi_{\mathrm{fd}} \tag{4}$$

$$\frac{V\sin\psi_f + \|W(x_f,t_f)\|_2\sin\theta_W(x_f,t_f)}{V\cos\psi_f + \|W(x_f,t_f)\|_2\cos\theta_W(x_f,t_f)} = \frac{\sin\psi_{\mathrm{fd}}}{\cos\psi_{\mathrm{fd}}} \tag{5}$$

$$V\sin(\psi_f - \psi_{\mathrm{fd}}) = \|W(x_f,t_f)\|_2 \sin(\psi_{\mathrm{fd}} - \theta_W(x_f,t_f)) \tag{6}$$

which leads to

$$\psi_f = \arcsin\left(\frac{\|W(x_f,t_f)\|_2}{V}\sin(\psi_{\mathrm{fd}} - \theta_W(x_f,t_f))\right) + \psi_{\mathrm{fd}} \tag{7}$$

where

$$\theta_W(x_f,t_f) = \arctan\frac{W_2(x_f,t_f)}{W_1(x_f,t_f)} \tag{8}$$

This leads us to Problem 2.

Problem 2: For the system (1), minimize t_f subject to the constraints: $q(0) = q_0 = (x_0, \psi_0)$; $x(t_f) = x_f$; and $\angle\dot{x}(t_f) = \psi_{\mathrm{fd}}$; $u(\cdot)$ is a Lebesgue measurable function, and satisfies Eq. (2).

In this paper, we will focus on the special case $W(x,t) = W(t)$. An existence of the minimum-time solution result for the more general case of time and space-varying winds (for Problem 1) can be found in McNeely [12]. In the same work, we proved the uniqueness of minimum-time solutions for almost every combination of initial and final states for Problem 1. Here, we prove existence and uniqueness of minimum-time solutions for the case of time-varying wind vector fields [that is, $W(x,t) = W(t)$] for Problem 2.

In other related work, Lissaman [13] and Zhao and Ying [14] discuss the use of wind gradients to do work for the vehicle while in flight. Another study by Venkataramanan and Dogan [15] is based upon the use of a nonlinear controller when reconfiguring the formation of a group of UAVs and how wind effects are taken into consideration. Studies by Boyle and Chamitoff [16] are based on a stepwise solution to a constrained optimization problem which solves a nonlinear two-point boundary value problem at each time step.

Tour Problem

In the considered mission, a small unmanned air vehicle (SUAV) gives the MAV a set of N targets $x_i = (x_{i1}, x_{i2})$; $i = 1, \ldots, N$ to fly over. The case in which the sensor is located on the aircraft, such that a straight line flight along the centerline must be altered to view the target, is outside the scope of this paper though it is an interesting problem in its own right. For each target, the street where it is located is found on a map which yields two possible angles ψ_i or $\psi_i + \pi$ for flying over the target. The midline of that street is found and the point x_i is projected perpendicularly from the target onto that line. To simplify notation, we will represent this new point also as x_i and continue to work with these coordinates henceforth. For the ith target, two way points (x_i^1, x_i^2) are created equally spaced along the street about the projected point. These are considered entrance/exit points for viewing the target. Thus, each exit point is described by coordinates $q_i^j = (x_{i1}^j, x_{i2}^j, \psi_i^j)$, where $i = 1, 2, \ldots, N$; $j = 1, 2$; and ψ_i^j is chosen to be one of the two angles ψ_i or $\psi_i + \pi$, such that the unit vector $(\cos\psi_i^j, \sin\psi_i^j)$ points away from the target (x_i, y_i) at the point x_i. The objective is to fly from target i to target $k \ne i$ by flying from a point q_i^j to q_k^l. Thus there are four possible paths corresponding to j, $l = 1, 2$. In this paper, we do not address the problem of flying between q_i^r to q_i^s (that is flying over the target x_i) where $r \ne s$, and assume it to be a simple straight line along the street. Once the cost of flying the optimal path between q_i^j to q_k^l for all $i, k = 1, 2, \ldots, N$; $i \ne k$ and $j, l = 1, 2$ then one can set up a task assignment problem as outlined in [3].

Discussion of Dubins's Theorem for the Zero-Wind Case

We briefly discuss Dubins's result on the existence and uniqueness of minimum length solutions for a particle following a Lipschitz continuous path at a constant speed V, with a bound on the maximum of the curvature given by $1/R_{\min}$. This theorem states that for every initial, final positions and velocities the minimum time (or minimum length) solution is an arc–line–arc or arc–arc–arc curve. Here, by "arc" we mean a portion of a circle or radius $R_{\min}$ and "line" means a straight line segment. Furthermore, for the arc–arc–arc solution, Dubins's theorem states that the middle arc must have length greater than $\pi R_{\min}$. For the tour planning problem, Dubins's theorem [8] posits the existence of a solution to the minimum time-optimal control problem for the special case $W(x,t) = (0,0)$ for all $x \in \mathrm{IR}^2$ and $t \in \mathrm{IR}_+$

As the minimum-time solution is invariant with respect to translations and rotations of the coordinate axis, we can change coordinates so that the initial position is at the origin of IR^2 and the final position is along the x axis. The initial and final velocity directions measured with respect to the x axis are termed ϕ_0 and ϕ_f, respectively, in Figs. 1 and 2. The direction of the velocity induces an orientation on the circles tangent to the velocity vector. In the figures,

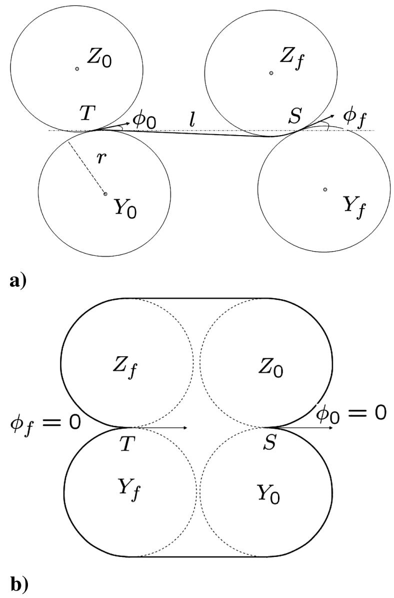


Fig. 1 Illustration of arc–line–arc solutions for the minimum time-optimal control problem. a) Illustration of an arc–line–arc Y_0LZ_f minimum length/time solution for sufficiently separated initial and final positions. b) Illustration of nonunique arc–line–arc minimum length solutions.

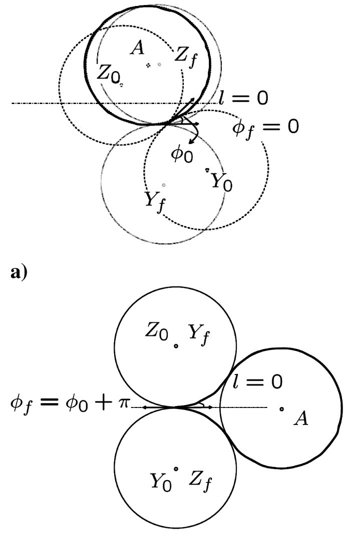


Fig. 2 Illustration of arc–arc–arc solutions for the minimum time-optimal control problem. a) Illustration of an arc–arc–arc solution when the initial and final positions coincide. b) Illustration of nonunique arc–arc–arc solutions.

we denote the center of the counterclockwise oriented circle by Z_0 and Z_f, respectively, while the centers of the clockwise oriented curves are denoted by Y_0 and Y_f, respectively. Thus we can distinguish between the Z_0LY_f from the Y_0LZ_f arc–line–arc solutions, etc. The arc–line–arc solution is found when the initial and final positions are fairly separated as in Fig. 1, whereas the arc–arc–arc solution is usually found when the initial and final positions are close enough as in Fig. 2. $l = 0$ in Fig. 2 means that the distance between the initial and final positions is zero. Figure 2a shows a $Y_0Z_fY_f$ arc–arc–arc curve connecting the initial and final velocities. Though the minimum-time solutions exist, they need not be unique as seen in Figs. 1b and 2b. A more important point from the point of view of numerical methods is the differentiability of the arclengths as a function of the final positions and velocities while holding the initial positions and velocities fixed. This issue is discussed in [12] in detail. The final result is that there is a set of final positions and velocities that has a measure zero, and where the minimum-time curves change from arc–line–arc to arc–arc–arc type or vice versa and at these points the arc length function is discontinuous and hence nondifferentiable. It is also possible for arc–line–arc curves to change to arc–line–arc curves in a discontinuous manner.

Time-Optimal Trajectories in the Presence of Wind

For a nonzero time-varying (Caratheodory) wind vector field, it is possible to show existence of a time-optimal solution if we make the following assumptions. Suppose that W is a differentiable function of t with bounded derivative and furthermore,

A1) $\|W\|_\infty = \sup_t \|W(t)\| < V$,

A2) $\|\partial W/\partial t\|_\infty \leq \beta \sqrt{V^2 - \|W\|_\infty^2}\, u_{\max}$,

where $\|\partial W/\partial t\|_\infty = \sup_t \|(\partial W/\partial t)(t)\|_2$, and $0 < \beta < 1$ is some constant.

Remarks: The assumption A1 is easy to explain. It is needed to ensure that the MAV will have nonzero forward velocity in the presence of the worst-case head wind. The assumption A2 is needed because the MAV has a limitation on the maximum rate of turn given by inequality (2) and hence cannot keep up with very rapid large changes in the wind direction (refer to McNeely [12] for more details).

Solution to Problem 1

In [12] we proved the following theorem (the statement has been modified from [12] to include two results proved in the same manuscript):

Theorem 1: Suppose that $W(t)$ is a time-varying (Carathéodory) vector field of wind velocities satisfying assumptions A1 and A2. Then there exists a solution to Problem 1 for the system (1). The solution is unique for almost every collection of initial and final states.

Suppose the wind $W(x,t) = W(t)$ depends only on the time variable, while satisfying the conditions A1 and A2. Then, we can the transform coordinates (with the same time variable) as follows:

$$\bar{x} = \varphi(x,t) = x - \int_0^t W(s)\,\mathrm{d}s \quad \text{and} \quad \bar{\psi} = \psi \tag{9}$$

Let $q(t) = (x(t), \psi(t))$ be the solution to Eq. (1) with initial condition q_0. After coordinate transformation, this solution becomes

$$\bar{x}(t) = x(t) - \int_0^t W(s)\,\mathrm{d}s \quad \text{and} \quad \bar{\psi}(t) = \psi(t) \tag{10}$$

Therefore, in the new coordinates $(\bar{x}, \bar{\psi})$, the system equations take on the form:

$$\dot{\bar{x}} = V(\cos\bar{\psi}, \sin\bar{\psi}), \qquad \bar{x}(0) = x_0 \tag{11}$$

$$\dot{\bar{\psi}} = u, \qquad \bar{\psi}(0) = \psi_0 \tag{12}$$

which is of the form considered by Dubins. The final states of the transformed system for Problem 1 are given by

$$\bar{x}(t_f) = x_f - \int_0^{t_f} W(s)\,\mathrm{d}s, \qquad \bar{\psi}(t_f) = \psi_{\mathrm{fd}} \tag{13}$$

By Theorem 1, the time-optimal solution exists for all initial and final conditions and is unique for almost every set of initial and final conditions for the original system. Clearly, the optimal trajectory must be a Dubins solution for the transformed system with initial states $(\bar{x}, \bar{\psi})(0)$ and final states $(\bar{x}, \bar{\psi})(t_f)$ (see McNeely [12] for a fuller discussion). The problem is that t_f is unknown and hence $\bar{x}(t_f)$ is also unknown. In the algorithm in Theorem 2, we make an initial guess for t_f and improve it using Newton's method. The convergence of the sequence of final times is proved.

Theorem 2: Select $\tau_f^0 > 0$. For $n = 1, 2, \ldots,$ consider the final states:

$$\bar{x}\left(\tau_f^{n-1}\right) = x_f - \int_0^{\tau_f^{n-1}} W(s)\,\mathrm{d}s; \qquad \bar{\psi}\left(\tau_f^{n-1}\right) = \psi_{\mathrm{fd}} \tag{14}$$

Let $L(\tau_f^{n-1})$ be the length of the minimum-time Dubins trajectory with initial state $(\bar{x}, \bar{\psi})(0)$ and final states $(\bar{x}, \bar{\psi})(\tau_f^{n-1})$, and denote

$$T\left(\tau_f^{n-1}\right) = \frac{L\left(\tau_f^{n-1}\right)}{V} \tag{15}$$

Set

$$\tau_f^n = \tau_f^{n-1} - \frac{\tau_f^{n-1} - T\left(\tau_f^{n-1}\right)}{1 - T'\left(\tau_f^{n-1}\right)} \tag{16}$$

Then, the sequence $\{\tau_f^n; n \in \mathbb{Z}\}$ converges to a unique $\tau_f^* > 0$ for almost every set of initial and final conditions for the original system provided τ_f^0 is sufficiently close to τ_f^*.

Proof: Consider the function $f(\tau) = \tau - T(\tau)$. The equation $f(\tau) = 0$ has a unique solution τ_f^* by Theorem 1 for almost every set of initial and final states. This is because $f(\tau_f^*) = 0$ yields $\tau_f^* = L(\tau_f^*)/V$. This means that the minimum-time trajectory from $\bar{x}(0)$ to the point $x_f - \int_0^{\tau_f^*} W(s)\,\mathrm{d}s$ takes exactly time τ_f^*, or, in other words, $\bar{x}(\tau_f^*) = x_f - \int_0^{\tau_f^*} W(s)\,\mathrm{d}s$. With the existence and uniqueness of τ_f^* having been addressed, let us consider the differentiability of the function $L(\tau)$. As studied by McNeely [12], this function is continuously differentiable for almost every choice of τ. Hence, provided the initial selection τ_f^0 is sufficiently close[17] to τ_f^*, Newton's method is applicable to the solution of $f(\tau) = 0$ which results in (16). □

Remark: Unfortunately, it is difficult to characterize how "close" the initial selection τ_f^0 needs to be to τ_f^* as this depends on x_0, x_f, ψ_0, ψ_f and the wind vector field $W(t)$ in a complicated way.

Solution to Problem 2

We will first prove the existence of time-optimal solutions for the special case of time-varying wind $W(x, t) = W(t)$. The proof for the general case also follows the same pattern though the equations look more complicated. As the algorithm in this paper is designed for the case for $W(x, t) = W(t)$, this is the significant case for our purposes. We have the following theorem.

Theorem 3: Suppose that $W(x, t) = W(t)$ is a time-varying (Carathéodory) vector field of wind velocities satisfying assumptions A1 and A2. Then there exists a solution to Problem 2 for the system (1), and the solution is unique for almost every collection of initial and final states.

Proof: *Existence*: The proof of existence proceeds in similar fashion to the proof of Theorem 1 as given in [12]. The idea is to establish one solution and then use Filippov's theorem [12,18] to prove the existence of a minimum-time solution. The worst-case wind vector field is the one that yields the system equations:

$$\dot{x} = (V + \|W\|_\infty)(\cos\psi, \sin\psi); \qquad \dot{\psi} = u \tag{17}$$

as the instantaneous turn radius given by

$$R'_{\min}(t) = \frac{\|\dot{x}\|}{\max\dot{\theta}} = \frac{V + \|W\|_\infty}{V} R_{\min}$$

is the largest possible. The condition on $\psi(t_f)$ becomes $\psi(t_f) = \psi_{\mathrm{fd}}$. Dubins's theorem [8] is applicable to system (17) with initial states (x_0, ψ_0) and final states $(x_f, \psi_{\mathrm{fd}})$, and it yields the existence of a minimum-time solution for the worst-case wind. For a given wind field $W(t)$ one can construct a piecewise continuous $u(t)$ such that the solution stays on the trajectory for the worst-case wind as shown in the proof of Theorem 1 in McNeely [12]. This solution satisfies $\angle\dot{x}(t_f) = \psi_{\mathrm{fd}}$ for some $t_f > 0$ that is exactly what was desired. However, this solution is not the minimum time-optimal solution for the wind $W(t)$, and we prove the existence of the latter using Filippov's theorem. To apply the theorem, we need to make a change of variables. Let

$$\alpha(t) = \arctan\left(\frac{V\sin\psi(t) + W_2(t)}{V\cos\psi(t) + W_1(t)}\right) \tag{18}$$

Then we obtain (the variable t is suppressed for brevity):

$$\dot{\alpha} = \frac{(V\cos\psi + W_1)(Vu\cos\psi + \dot{W}_2) - (V\sin\psi + W_2)(-Vu\sin\psi + \dot{W}_1)}{V^2 + \|W\|^2 + 2V(W_2\sin\psi + W_1\cos\psi)} \tag{19}$$

$$= \frac{V^2u + Vu(W_2\sin\psi + W_1\cos\psi) + V(\dot{W}_2\cos\psi - \dot{W}_1\sin\psi) + (W_1\dot{W}_2 - W_2\dot{W}_1)}{V^2 + \|W\|^2 + 2V(W_2\sin\psi + W_1\cos\psi)} \tag{20}$$

The above equation can be written entirely in terms of $\alpha(t)$ using the equations:

$$\psi(t) = \arcsin\left(\frac{\|W(t)\|_2}{V}\sin(\alpha(t) - \theta_W(t))\right) + \alpha(t)$$
$$\theta_W(t) = \arctan\frac{W_2(t)}{W_1(t)} \tag{21}$$

The system in variables (x, α) has initial and final conditions as follows:

$$x(0) = x_0; \qquad x(t_f) = x_f \tag{22}$$

$$\alpha(0) = \arctan\left(\frac{V\sin\psi_0 + W_2(0)}{V\cos\psi_0 + W_1(0)}\right); \qquad \alpha(t_f) = \psi_{\mathrm{fd}} \tag{23}$$

It is straightforward to see that the system in the variables (x, α) satisfies all the conditions of Filippov's theorem [12,18]. As we have already shown the existence of one solution that satisfies the initial and final conditions, there exists a measurable control $u^*(\cdot)$ and a minimum-time solution $x^*(\cdot)$, $\alpha^*(\cdot)$ that satisfies the boundary conditions for almost all initial and final states.

Uniqueness: To show the uniqueness of the solution for almost all initial and final states, we need to work with the $\bar{x}$, $\bar{\psi}$ coordinates as in Eq. (9):

$$\dot{\bar{x}} = V(\cos\bar{\psi}, \sin\bar{\psi}), \qquad \bar{x}(0) = x_0$$
$$\bar{x}(t_f) = x_f - \int_0^{t_f} W(s)\,\mathrm{d}s \tag{24}$$

$$\dot{\bar{\psi}} = u, \qquad \bar{\psi}(0) = \psi_0$$
$$\arctan\left(\frac{V \sin\bar{\psi}(t_f) + W_2(t_f)}{V \cos\bar{\psi}(t_f) + W_1(t_f)}\right) = \psi_{\mathrm{fd}} \tag{25}$$

By our proof of existence, we know that there exists a minimum-time solution for almost all initial and final states for the original system. Assuming it exists, let $(x, \psi)(t)$, $t \in [0, t_f]$ be this solution. Denoting $\bar{x}(t_f)$ by $\bar{x}_f$ and $\bar{\psi}(t_f)$ by $\bar{\psi}_f$, the final states in Eqs. (24) and (25) can be written as

$$\bar{x}_f = x_f - \int_0^{t_f} W(s)\,\mathrm{d}s \tag{26}$$

$$\bar{\psi}_f = \arcsin\left(\frac{\|W(t_f)\|_2}{V}\sin(\psi_{\mathrm{fd}} - \theta_W(t_f))\right) + \psi_{\mathrm{fd}}$$
$$\theta_W(t_f) = \arctan\frac{W_2(t_f)}{W_1(t_f)} \tag{27}$$

With $(\bar{x}_f, \bar{\psi}_f)$ as the desired final state for the transformed system, we see that Dubins's theorem [8] applies and there exists a unique solution $(\bar{x}', \bar{\psi}')(\cdot)$ defined on $[0, t_f^*]$ for almost every combination of initial and final states of the transformed system, where $t_f^* \leq t_f$. We can transform the solution back to the original coordinates (x', ψ') while remembering that

$$\bar{x}'(t_f^*) = \bar{x}_f, \qquad \bar{\psi}'(t_f^*) = \bar{\psi}_f \tag{28}$$

to obtain

$$x'(t_f^*) = \bar{x}'(t_f^*) + \int_0^{t_f^*} W(s)\,\mathrm{d}s \tag{29}$$

$$= x_f + \int_{t_f}^{t_f^*} W(s)\,\mathrm{d}s \tag{30}$$

$$\psi'(t_f^*) = \psi_f \tag{31}$$

A sufficient condition for $x'(t_f^*) = x_f$ and $\psi'(t_f^*) = \psi_f$ is $t_f^* = t_f$. If $t_f^* < t_f$, then it follows that both solutions $(x, \psi)(t)$, $t \in [0, t_f]$ and $(x', \psi')(t)$, $t \in [0, t_f^*]$ satisfy the boundary conditions, and the first solution is not the minimum-time solution contradicting our original assumption. Hence $t_f^* = t_f$ is both necessary and sufficient for $x'(t_f^*) = x_f$ and $\psi'(t_f^*) = \psi_f$. As the Dubins solution in the transformed coordinates is unique for almost every choice of end conditions, the same is true for the minimum-time solution in the original coordinates as well. □

The same algorithm and convergence proof for Problem 1 as summarized in Theorem 2 also applies to Problem 2 with a minor change.

Theorem 4: Select $\tau_f^0 > 0$. For $n = 1, 2, \ldots$, consider the final states:

$$\bar{x}\left(\tau_f^{n-1}\right) = x_f - \int_0^{\tau_f^{n-1}} W(s)\,\mathrm{d}s \tag{32}$$

$$\bar{\psi}\left(\tau_f^{n-1}\right) = \arcsin\left(\frac{\left\|W\left(\tau_f^{n-1}\right)\right\|_2}{V}\sin\left(\psi_{\mathrm{fd}} - \theta_W\left(\tau_f^{n-1}\right)\right)\right) + \psi_{\mathrm{fd}} \tag{33}$$

where

$$\theta_W\left(\tau_f^{n-1}\right) = \arctan\frac{W_2\left(\tau_f^{n-1}\right)}{W_1\left(\tau_f^{n-1}\right)} \tag{34}$$

Let $L(\tau_f^{n-1})$ be the length of the minimum-time Dubins trajectory with initial state $(\bar{x}, \bar{\psi})(0)$ and final states $(\bar{x}, \bar{\psi})(\tau_f^{n-1})$ and denote

$$T\left(\tau_f^{n-1}\right) = \frac{L\left(\tau_f^{n-1}\right)}{V} \tag{35}$$

Set

$$\tau_f^n = \tau_f^{n-1} - \frac{\tau_f^{n-1} - T\left(\tau_f^{n-1}\right)}{1 - T'\left(\tau_f^{n-1}\right)} \tag{36}$$

Then, the sequence $\{\tau_f^n; n \in \mathbb{Z}\}$ converges to a unique $\tau_f^* > 0$ for almost every set of initial and final conditions for the original system provided τ_f^0 is sufficiently close to τ_f^*.

Proof: The proof is identical to that of Theorem 2, except for the fact that Theorem 3 is invoked in the proof instead of Theorem 1. □

Next, we describe the algorithms given in Theorem 2 for Problem 1, and Theorem 4 for Problem 2 in greater detail. In Step 2, we change coordinates for numerical stability from (x, ψ) to (z, θ) such that in the new coordinates z_0 is at the origin and z_f is along the positive z_1 axis. One could also employ a scaling of the axes in this step. In Step 3, we set the wind to zero and use Dubins's method to obtain an initial estimate τ_f^0 for the final time. In Step 4, we modify the final angle θ_{fd} to θ_f for Problem 2. This step is not necessary for Problem 1. At Step n, the estimate to t_f is τ_f^n, and yields θ_f^n. On convergence of the algorithm the correct value for θ_f will be found along with t_f.

Algorithm for Time-Optimal Paths in the Presence of Time-Varying Wind

1) *Start*

Input $q_0 = (x_{10}, x_{20}, \psi_0)$ and $q_f = (x_{1f}, x_{2f}, \psi_{\mathrm{fd}})$ (ψ_{fd} is the desired final angle), minimum turn radius ($R_{\min}$), speed (V), and time-varying wind velocity $[W(t)]$. Select a tolerance $\epsilon > 0$.

2) *Perform a coordinate transformation for numerical stability*

Let $\cos\phi = [(x_{1f} - x_{10})/\|x_f - x_0\|]$ and $\sin\phi = [(x_{2f} - x_{20})/\|x_f - x_0\|]$.

Define

$$R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

We use the rotation matrix R and a shift of origin so that (x, ψ) is transformed into (z, θ), with z_f along the positive z_1 axis. Specifically, $z = R^T(x - x_0)$, and $\theta = \psi - \phi$. We have to consider four cases to compute ϕ in an actual implementation:

a) If $\cos\phi = 0$ and $\sin\phi > 0$, let $\phi = \pi/2$.
b) If $\cos\phi = 0$ and $\sin\phi < 0$, let $\phi = -\pi/2$.
c) If $\cos\phi > 0$, let $\phi = \arctan(\sin\phi/\cos\phi)$.
d) Else $\phi = \arctan(\sin\phi/\cos\phi) + \pi$.

Denote $\theta_0 = \psi_0 - \phi$ and $\theta_{\mathrm{fd}} = \psi_{\mathrm{fd}} - \phi$.

z_0 and z_f are computed according to the coordinate transform $z_0 = (0, 0)$; $z_f = R^T(x_f - x_0)$. The wind vector field $W(t)$ is transformed to $\bar{W}(t) = R^T W(t)$. The new system of equations is

$$\dot{z} = V(\cos\theta, \sin\theta) + \bar{W}(t); \qquad \dot{\theta} = u \tag{37}$$

3) *Step 0: Use Dubins's method to find an initial value for τ_f^0.*

Use (z_0, θ_0) and (z_f, θ_f) to compute a Dubins trajectory for the transformed system with zero wind. Let L be the length of this trajectory. Set $\tau_f^0 = \frac{L}{V}$.

4) *Step n: Find the angle θ_{fd}.*

Find θ_f dependent on θ_{fd}, V, and $\bar{W}(\tau_f^n)$. Figure 3 describes the idea behind the computation.

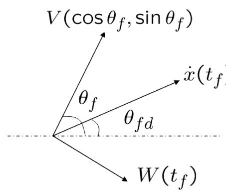


Fig. 3 Relationship of θ_{fd} and θ_f.

Denote by $\bar{V} = (\bar{V}_1, \bar{V}_2)$ the vector $V(\cos\theta_f, \sin\theta_f)$. At $t = \tau_f^n$, Consider $\bar{V} + \bar{W}(\tau_f^n) = (\bar{V}_1 + \bar{W}_1(\tau_f^n), \bar{V}_2 + \bar{W}_2(\tau_f^n))$, where $\bar{V}$ is as yet unknown (because τ_f^n is unknown).

Note that $\tan\theta_{fd} = [(\bar{V}_2 + \bar{W}_2(\tau_f^n))/(\bar{V}_1 + \bar{W}_1(\tau_f^n))]$.

Therefore

$$\bar{V}_1 + \bar{W}_1(\tau_f^n) = \|\bar{V} + \bar{W}(\tau_f^n)\| \cos\theta_{fd} \tag{38}$$

$$\bar{V}_2 + \bar{W}_2(\tau_f^n) = \|\bar{V} + \bar{W}(\tau_f^n)\| \sin\theta_{fd} \tag{39}$$

If we combine these equations and solve for $\bar{V}_2$, we have the following equation:

$$\bar{V}_2 = (\bar{V}_1 + \bar{W}_1(\tau_f^n)) \tan\theta_{fd} - \bar{W}_2(\tau_f^n) \tag{40}$$

We also have

$$\bar{V}_1^2 + \bar{V}_2^2 = V^2 \tag{41}$$

Solve for $\bar{V}_1$ and $\bar{V}_2$ using Eqs. (40) and (41).

As $\cos(\theta_f) = \bar{V}_1/V$ and $\sin(\theta_f) = \bar{V}_2/V$, we find θ_f using the following process.

a) If $\cos\theta_f = 0$ and $\sin\theta_f > 0$, let $\theta_f = \pi/2$.
b) If $\cos\theta_f = 0$ and $\sin\theta_f < 0$, let $\theta_f = -\pi/2$.
c) If $\cos\theta_f > 0$, let $\theta_f = \arctan(\sin\theta_f / \cos\theta_f)$.
d) Else $\theta_f = \arctan(\sin\theta_f / \cos\theta_f) + \pi$.

5) *Step n continued: compute* τ_f^n:

Let $\bar{z}_f = z_f - \int_0^{\tau_f^{(n-1)}} \bar{W}(s)\,\mathrm{d}s$. Use Dubins's method with initial state (z_0, θ_0) and final state $(\bar{z}_f, \theta_f^{n-1})$. Denote the length of the path divided by the speed V by $T(\tau_f^{(n-1)})$ as it is a function of $\tau_f^{(n-1)}$.

6) If $|T - \tau_f^{(n-1)}| < \epsilon$, then go to Item 7. Otherwise, compute the solution τ_f^n to $f(\tau) = T(\tau) - \tau$ using Newton's method or modified Newton's method [17]. $\tau_f^n = \tau_f^{n-1} - [f(\tau_f^{n-1})/f'(\tau_f^{n-1})]$. The derivative term in the denominator is computed numerically. Go to Item 4.

7) *Stop*. We take τ_f^n to be the approximate value of τ_f^*.

Remark: For the case of constant wind $W(t) = W$, the algorithm is particularly simple to implement and intuitive. It can be interpreted as follows. We seek to find a state $(\bar{x}_f^w, \bar{\psi}_f)$ such that in the absence of wind, the Dubins's solution takes time t_f to reach this state from $(\bar{x}_0, \bar{\psi}_0)$. The property of this state is that it takes exactly time t_f for a particle to go from $\bar{x}_f^w$ to $\bar{x}_f$, with speed W along a straight line. The same interpretation also holds for the case of time-varying wind.

Numerical Results

To illustrate the effectiveness of the algorithm, we look at a few numerical examples while considering a constant wind vector field. In practice, it is not possible to have knowledge of the wind vector field $W(t)$ before hand, and hence constant wind is the important case to consider. We first consider a simple trajectory planning problem with given initial and desired final positions x_0, x_f and headings ψ_0, ψ_{fd}. We consider Problem 2 for several cases with various constant wind vector fields. For all cases, we set $V = 40$ m/s and $R_{\min} = 100$ m. The units of distance in the discussion below is meters, that of speed is meters per second, and that of angle is radians.

Table 1 Wind vectors and minimum times for optimal trajectories

Wind vector, m/s	Time, s
[−10, −15]	23.33
[−5, −10]	19.75
[0, 0]	16.61
[5, 5]	17.22
[10, 10]	20.25
[10, 15]	19.11

Trajectory Planning Example

We consider two generic initial and final states, where the final states were chosen using the random number generator in MATLAB. The wind vectors range from $W = [-10; -15]$ to $W = [10; 15]$ and all satisfy the condition $\|W\|_2 < V$. Specifically, the states were $q_0 = (-650, -100, \pi/4)$ and $q_f = (-692.9, 433.62, -0.1824)$. The tolerance ϵ for the algorithm was chosen to be $\epsilon = 0.1$. Applying the algorithm for Problem 2, the results shown in Fig. 4 and summarized in Table 1 were obtained. Figure 4 shows the actual trajectories computed and their variation with the wind can be seen. Notice that all trajectories are tangential to each other at the endpoint in spite of the different wind conditions, which is exactly as required for solutions to Problem 2. As the wind increases from $W = [10; 10]$ m/s to $[10; 15]$ m/s, a qualitative change in the trajectory can be observed. This could perhaps be explained by observing that the minimum-time trajectory would be the one with the maximum "coasting" in the direction of the wind.

Next, we consider a case where the initial and final positions are identical—to be precise, $q_0 = (0, 0, 0)$ and $q_f = (0, 0, \pi)$. It is well known that this problem has a nonunique arc–arc–arc solution for the zero-wind case and the algorithm computes one of the solutions (the clockwise one) as shown in Fig. 5. The same figure also shows the solution to Problem 2 in the presence of constant wind

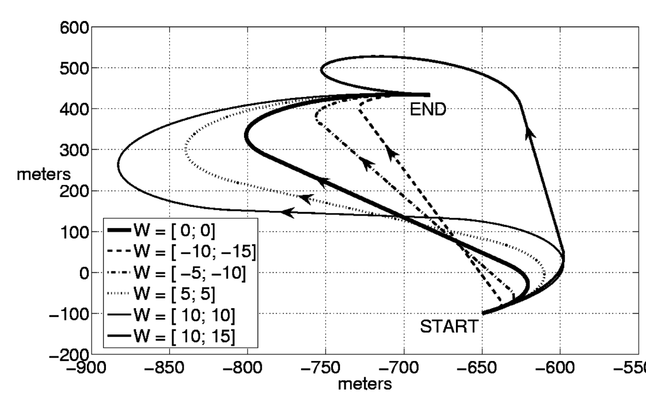


Fig. 4 Minimum-time arc–line–arc solutions for zero and nonzero wind conditions.

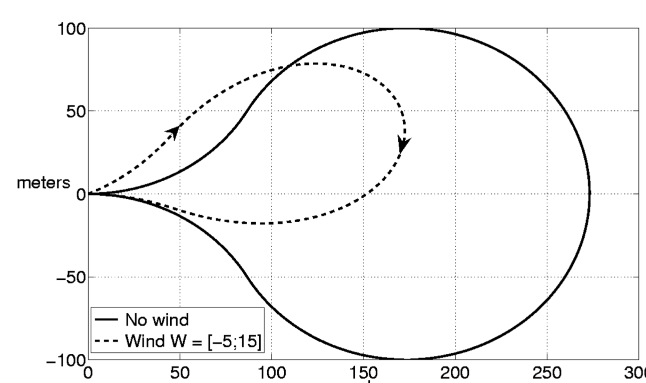
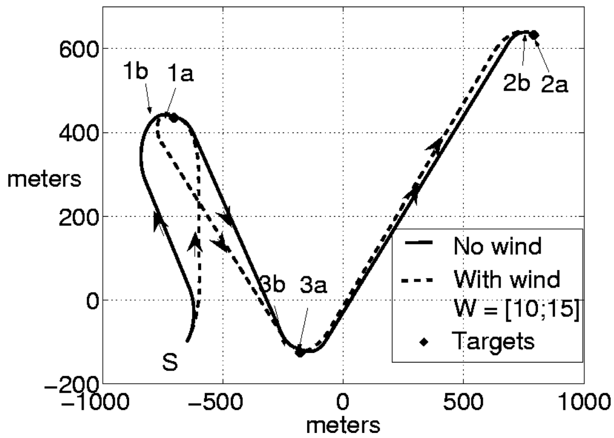
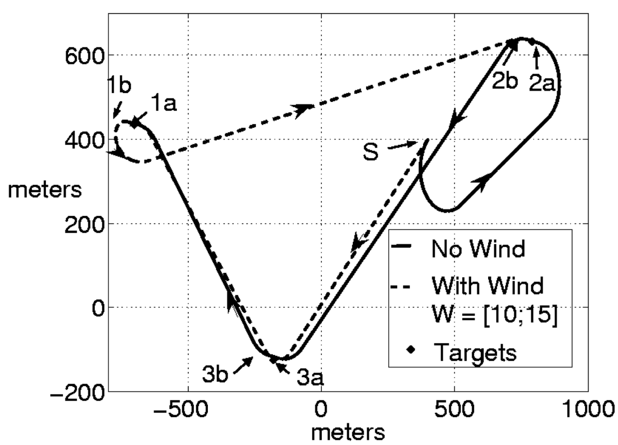


Fig. 5 Minimum-time arc–arc–arc solutions for zero and nonzero wind conditions.



a)



b)

Fig. 6 Minimum path examples comparing wind and no wind tours of three targets. a) Tour planning problem 1; b) tour planning problem 2.

$W = [-5; 15]$ m/s. Notice that now the symmetry is lost, and the solution to Problem 2 is a clockwise trajectory which is tangential to the x axis at the final time, as required.

Optimal Tour Planning

In this section, we consider the effect of the wind on the tour planning or task assignment for MAVs. Given several target locations, an initial starting point, and heading, we would like to find the order in which to visit the targets such that the total time of travel is minimized. To do this, we first compute exit/entry points for each target as described in the section on the Tour Problem. Then we set up a directed graph containing all possible paths between the targets themselves, and between the starting state and the targets. The minimum time taken for each state transition is computed using the algorithm and assigned as the cost of the state transition. Finally, one needs to search the cost table for the directed graph to find a state transition sequence of minimum cost that starts with the initial state and covers all the targets. Several publications [1,2,9] in the literature have addressed the search problem and that is not the topic under investigation in this paper. As our interest is in demonstrating the effect of wind on the task assignment problem, we consider two examples with three targets each and the direct graph search was done manually with some effort. The wind for both examples was set at $[10; 15]$ m/s though this is just a number and the theory developed earlier does not depend on any specific wind speeds.

In the first example, the drop-off point is $(-650, -100, \frac{\pi}{4})$, and in the second example, the drop-off point is $(400, 400, \frac{5\pi}{4})$, and they have been denoted by S in Figs. 6a and 6b. The three targets are target 1 $\equiv (-703, 434.9)$, target 2 $\equiv (790.9, 632.2)$, and target 3 $\equiv (-179.5, -124.8)$. Note that these targets were chosen using the map of a simulated city that is available in the MultiUAV simulation [3].

The entry/exit points for each of the targets were found as discussed in the section on the Tour Problem. Those about the first target are $1a \equiv (-682.9, 433.6, -0.18)$ and $1b \equiv (-722.2, 440.9, 2.96)$. The second target has bounds $2a \equiv (810.8, 629.8, -0.19)$ and $2b \equiv (771.53, 637.19, 2.9565)$. The third target has entry/exit points $3a \equiv (-158.5, -121.0, -0.18)$ and $3c \equiv (-197.8, -113.7, 2.96)$. For the first example, Tables 2 and 3 were obtained for the state transition times, where the time is measured in seconds. The time-optimal tours are plotted in Fig. 6a. The results of the minimum-time tour computation showed an improvement in the time needed to travel the minimal path in these wind conditions. The optimal path where there is no wind present was computed to be S-$1b$-$1a$-$3b$-$3a$-$2b$-$2a$, which is travel from the drop-off point to the first target, then the third target, and finally to the second target. The cost of traveling this path in the case of no wind was 69.4 s. In the presence of wind, the optimal path was found to be S-$1a$-$1b$-$3b$-$3a$-$2b$-$2a$ and the cost got reduced to 64.9 s. Although travel to each target was performed in the same order, the approach was different in each scenario. This case shows how the presence of wind could help the trajectory of an MAV.

For the second example, only the relationships with S and the other points change. The cost tables for this problem are given in Tables 4

Table 2 Tour example 1: zero-wind case

	S	$1a$	$1b$	$2a$	$2b$	$3a$	$3b$
S	——	14.6	17.3	45.8	40.2	21.4	11.6
$1a$	——	——	1	44.3	36.7	24.5	18.6
$1b$	——	1	——	52.6	44.3	32.2	24.5
$2a$	——	44.3	52.6	——	1	35.0	40.6
$2b$	——	36.7	44.3	1	——	30.6	35.0
$3a$	——	24.5	32.2	35.0	30.6	——	1
$3b$	——	18.6	24.5	40.6	35.0	1	——

Table 3 Tour example 2: wind case

	S	$1a$	$1b$	$2a$	$2b$	$3a$	$3b$
S	——	11.5	19.6	35.9	29.5	15.6	10.6
$1a$	——	——	0.9	35.4	29.6	26.8	24.4
$1b$	——	1.2	——	43.0	37.2	31.5	28.3
$2a$	——	66.3	72.7	——	0.9	60.7	65.5
$2b$	——	57.4	63.6	1.2	——	54.4	58.5
$3a$	——	29.0	37.0	27.0	21.8	——	0.9
$3b$	——	18.1	26.6	32.1	25.6	1.2	——

Table 4 Tour example 2: zero-wind case

	S	$1a$	$1b$	$2a$	$2b$	$3a$	$3b$
S	——	27.3	36.1	22.8	19.8	19.5	23.9
$1a$	——	——	1	44.3	36.7	24.5	18.6
$1b$	——	1	——	52.6	44.3	32.2	24.5
$2a$	——	44.3	52.6	——	1	35.0	40.6
$2b$	——	36.7	44.3	1	——	30.6	35.0
$3a$	——	24.5	32.2	35.0	30.6	——	1
$3b$	——	18.6	24.5	40.6	35.0	1	——

Table 5 Tour example 2: wind case

	S	$1a$	$1b$	$2a$	$2b$	$3a$	$3b$
S	——	39.6	46.4	19.1	18.3	34.5	38.6
$1a$	——	——	0.9	35.4	29.6	26.8	24.4
$1b$	——	1.2	——	43.0	37.2	31.5	28.3
$2a$	——	66.3	72.7	——	0.9	60.7	65.5
$2b$	——	57.4	63.6	1.2	——	54.4	58.5
$3a$	——	29.0	37.0	27.0	21.8	——	0.9
$3b$	——	18.1	26.6	32.1	25.6	1.2	——

and 5. The time-optimal tours for the zero wind and wind case are shown in Fig. 6b. The results of the search routine showed that the minimal path in wind conditions was longer than that in no wind. The optimal path where there is no wind was S-$2a$-$2b$-$3a$-$3b$-$1a$-$1b$ with a cost of 75 s. In the presence of wind, the optimal path was S-$3a$-$3b$-$1a$-$1b$-$2b$-$2a$ and the cost was 92.8 s. This path is very different from that where no wind is present.

Conclusions

In this paper, we consider the problem of constructing minimum-time trajectories for a Dubins vehicle in the presence of a time-varying wind vector field. We have presented results on the existence and uniqueness of minimum-time solutions for a Dubins vehicle flying in a general time-varying wind vector field under some technical conditions. Using these results we have proposed and proved the convergence of an algorithm for obtaining a minimum-time solution for such a vehicle. We have presented the results of numerical experiments that show that the importance of considering wind vector fields while planning the tour for a Dubins vehicle.

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References

[1] Chandler, P., and Pachter, M., "Hierachical Control for Autonomous Teams," AIAA Paper 2001-4149, Aug. 2001.

[2] Schumacher, C., Chandler, P., and Rasmussen, S., "Task Allocation for Wide Area Search Munitions via Network Flow Optimization," AIAA Paper 2001-4147, Aug. 2001.

[3] Rasmussen, S., Mitchell, J. W., Chandler, P., Schumacher, C., and Smith, A. L., "Introduction to the MultiUAV2 Simulation and Its Application to Cooperative Control Research," *Proceedings of the American Control Conference*, Vol. 7, IEEE, Piscataway, NJ, June 2005, pp. 4490–4501.

[4] Chaudhry, A. I., Misovec, K. M., and D'Andrea, R., "Low Observability Path Planning for an Unmanned Air Vehicle Using Mixed Integer Linear Programming," *Proceedings of the 43rd IEEE Conference on Decision and Control*, Vol. 4, IEEE, Piscataway, NJ, Dec. 2004, pp. 3823–3829.

[5] Yang, G., and Kapila, V., "Optimal Path Planning for Unmanned Air Vehicles with Kinematic and Tactical Constraints," *Proceedings of the 41st IEEE Conference on Decision and Control*, Vol. 2, IEEE, Piscataway, NJ, Dec. 2002, pp. 1301–1306.

[6] Howlett, J., "Path Planning for Sensing Multiple Targets from an Aircraft," M.S. Thesis, Department of Mechanical Engineering, Brigham Young University, Provo, UT, Dec. 2002.

[7] Anderson, E., "Extremal Control and Unmanned Air Vehicle Trajectory Generation," M.S. Thesis, Department of Electrical and Computer Engineering, Brigham Young University, Provo, UT, April 2002.

[8] Dubins, L. E., "On Curves of Minimal Length with a Constraint on Average Curvature and with Prescribed Initial and Terminal Positions and Tangents," *American Journal of Mathematics*, Vol. 79, 1954, pp. 497–516.

[9] Savla, K., Bullo, F., and Frazzoli, E., "On Traveling Salesperson Problems for Dubins' Vehicle: Stochastic and Dynamic Environments," *Proceedings of the IEEE Conference on Decision and Control*, IEEE, Piscataway, NJ, Dec. 2005, pp. 4530–4535.

[10] Etkin, B., *Dynamics of Atmospheric Flight*, Wiley, New York, 1972, pp. 143–145.

[11] McGee, T., Spry, S., and Hedrick, K., "Optimal Path Planning in a Constant Wind with a Bounded Turning Rate," *AIAA Guidance, Navigation and Control*, AIAA, Reston, VA, 2006.

[12] McNeely, R., "Trajectory Planning for Micro Air Vehicles in the Presence of Wind," M.S. Thesis, Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX, May 2006.

[13] Lissaman, P., "Wind Energy Extraction by Birds and Flight Vehicles," AIAA Paper 2005-241, Jan. 2005.

[14] Zhao, Y., and Ying, Q., "Maximizing Endurance of Unmanned Aerial Vehicle Flight By Utilizing Wind Gradient," AIAA Paper 2003-6650, Sept. 2003.

[15] Venkataramanan, S., and Dogan, A., "Nonlinear Control for Reconfiguration of UAV Formation," AIAA Paper 2003-5725, Aug. 2003.

[16] Boyle, D. P., and Chamitoff, G. E., "Robust Nonlinear LASSO Control: A New Approach for Autonomous Trajectory Tracking," AIAA Paper 2003-5518, Aug. 2003.

[17] Stoer, J., and Bulirsch, R., *Introduction to Numerical Analysis*, 2nd ed., Springer–Verlag, New York, 1993, pp. 302–315.

[18] Filippov, A. F., "On Certain Questions in the Theory of Optimal Control," *SIAM Journal of Control, Series A*, Vol. 1, No. 1, 1962, pp. 76–84.